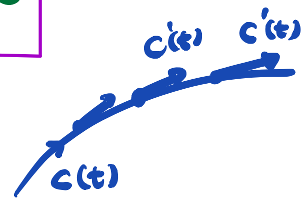
§ Geodesics & Exponential Map

<u>Idea</u>: geodesics = "straight lines" in a curved space (M,g).

"length-minimizing" ? "zero acceleration" ?

<u>Def</u>: A curve $c: I \to M$ is a geodesic in (M,g) if

$c'(t)$ is parallel along c, i.e. $\boxed{\dfrac{Dc'}{dt} = \nabla_{c'} c' \equiv 0}$ — (*)



Locally, (*) can be expressed as geodesic eqⁿ

$$\underbrace{\frac{d^2 c_k}{dt^2}(t) + \sum_{i,j} \Gamma_{ij}^k (c(t))\, c_i'(t)\, c_j'(t) = 0}_{\text{2nd order NON-Linear ODE system}}, \quad \forall k = 1, \dots, n$$

$\xRightarrow[\text{theory}]{\text{ODE}}$ short-time existence & uniqueness with initial data: $c(0)$, $c'(0)$

<u>Note</u>: Since c' is parallel, $g(c'(t), c'(t)) \equiv$ const.

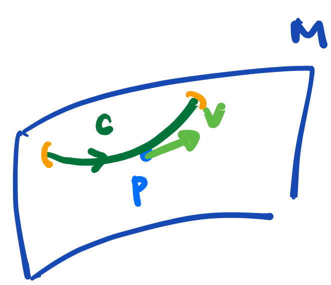
c is p.b.a.l. means $g(c'(t), c'(t)) \equiv 1$

<u>Thm</u>: Given $p \in M$, and $v \in T_pM$, $\exists$ (unique) smooth curve

$$c_{p,v}(t) : (-\varepsilon, \varepsilon) \longrightarrow M$$

s.t. $c_{p,v}$ is a geodesic on M with

$$c_{p,v}(0) = p \qquad c'_{p,v}(0) = v$$



Moreover, the curve $c_{p,v}$ depends smoothly on the initial data p and v. and the interval of existence ε

(homogenity)

Prop: $C_{p,\lambda v}(t) = C_{p,v}(\lambda t)$ for any $\lambda > 0$

(whenever the solutions are defined)

Proof: $\frac{d}{dt}(C_{p,v}(\lambda t)) = \lambda C'_{p,v}(\lambda t)$

$$\frac{d^2}{dt^2}(C_{p,v}(\lambda t)) = \lambda^2 C''_{p,v}(\lambda t)$$

Check: $C_{p,v}(0) = p$ & $\left.\frac{d}{dt}\right|_{t=0}(C_{p,v}(\lambda t)) = \lambda v$

and $(*)$ is satisfied.

$\square$

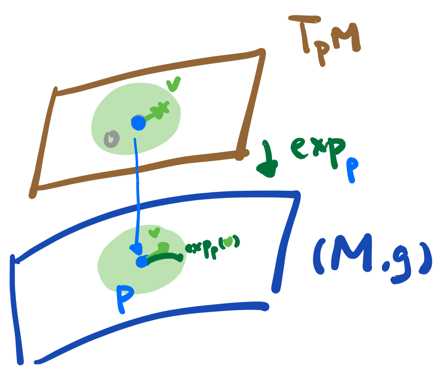
This implies that for any $p \in M$, $\exists$ nbd $\mathcal{U}$ of $0 \in T_pM$ s.t.

$$C_{p,v}(t) \text{ is defined } \forall t \in (-2, 2)$$

Def^{n}: The exponential map of (M, g) at p is

$$\exp_p : \overset{0 \in}{\mathcal{U}} \subseteq T_pM \longrightarrow M$$

$$\exp_p(v) := C_{p,v}(1)$$



Prop: $\exp_p$ is a local diffeo. at $0 \in T_pM$

Proof: Smooth dependence of $C_{p,v}$ on v $\Rightarrow$ $\exp_p$ smooth.

Clearly, $\exp_p(0) := C_{p,0}(1) = p$

Claim: $(d\exp_p)_0 = id_{T_pM} : T_pM \to T_pM$ (Note: $T_0(T_pM) \cong T_pM$)

$$(d\exp_p)_0(v) := \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \left.\frac{d}{dt}\right|_{t=0} C_{p,tv}(1)$$

$$\overset{\text{homogenity}}{=} \left.\frac{d}{dt}\right|_{t=0} C_{p,v}(t) = v$$

By I.V.F. Prop follows. $\square$

More generally, we can consider the exponential map

$$\begin{array}{rccc} \exp : & \tilde{\mathcal{U}} \subseteq TM & \longrightarrow & M \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (p, v) & \longmapsto & \exp_p(v) \end{array}$$

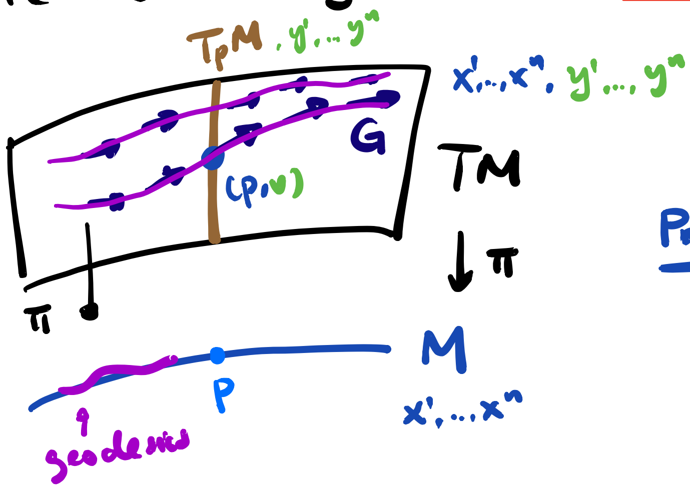
We can view the geodesic eq.^{n} (*) as a 1^{st} order ODE system at the level of tangent bundle:

<u>locally</u>: $TM \ni (p, v) \approx (\underbrace{x^1, \dots, x^n}_{p}, \underbrace{y^1, \dots, y^n}_{v = \sum_j y^j \frac{\partial}{\partial x^j}})$

$$(*) \iff \begin{cases} \dfrac{dx^k}{dt} = y^k \\[2ex] \dfrac{dy^k}{dt} = -\displaystyle\sum_{i,j} \Gamma_{ij}^k(x)\, y^i y^j \end{cases}$$

(R.H.S. is indep of t!)

i.e the R.H.S. gives us a <u>time-indep</u> vector field G on TM.



generates

geodesic flow $\{\varphi_t\} \subset \mathrm{Diff}(TM)$

<u>Prop</u>: The integral curves for this flow projects down to geodesics on M.

<u>Ex</u>: Prove this.

<u>Example 1</u>: $(M^n, g) = (\mathbb{R}^n, g_{Eucl.})$

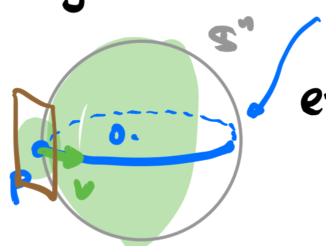
geodesics = straight lines (w/. const speed)

$$\exp_p(v) = p + v$$

$$c_{p,v}(t) = p + tv$$

<u>Example 2</u>: $(M^n, g) \cong (S^n, g_{round})$

geodesics = "great circles"



$\exp_p : B_\pi(0) \subseteq T_pM \to S^n \setminus \{-p\}$ is diffeomorphism.

Here is a recap of what we have discussed so far:

Recall: A Riemannian manifold (M^n, g)

where M^n = smooth n-dim'l manifold

For $p \in M$, $g_p := \langle \, , \, \rangle_p$ inner product on the vector space T_pM

$\hookrightarrow$ concept of "length" & "angles" on each T_pM

Thm: Given (M^n, g), $\exists !$ Levi-Civita connection ∇ s.t.

(1) $X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$

where $X, Y, Z \in \Gamma(TM)$

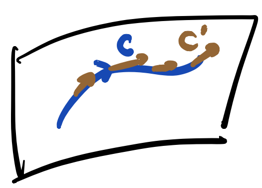
In local coord. $(x^1, \dots, x^n)$ on M, write $\partial_i := \frac{\partial}{\partial x^i}$

$g_{ij}^{(p)} = \langle \partial_i, \partial_j \rangle_p$, (g_{ij}) : $n \times n$ symm pos. definite matrix $\rightsquigarrow$ inverse matrix (g^{ij})

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k \overset{(1) \& (2)}{\rightsquigarrow} \Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

Note: $\Gamma_{ij}^k = F(g, \partial g)$.

Geodesic eq^n: $\nabla_{c'} c' \equiv 0$

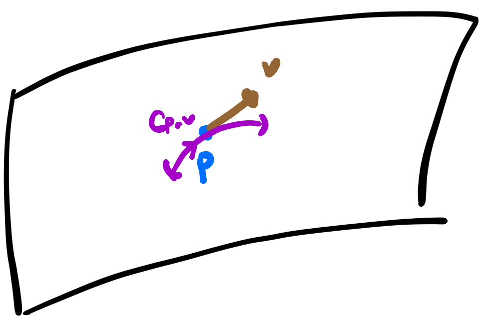


(M,g)

local coord: $\frac{d^2 c^k}{dt^2} + \Gamma_{ij}^k(c(t)) \frac{dc^i}{dt} \frac{dc^j}{dt} = 0$

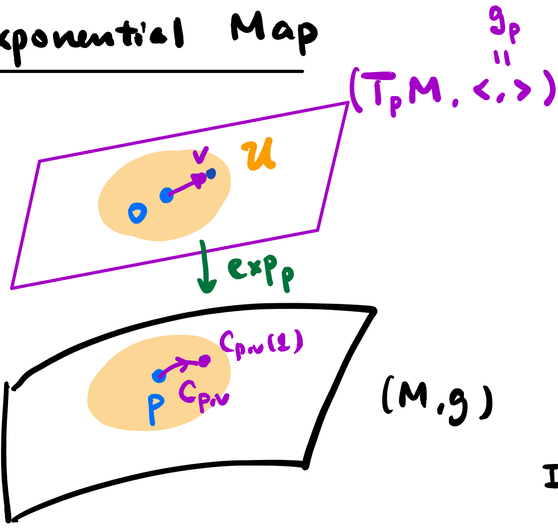
$\xrightarrow[\text{theory}]{\text{ODE}}$ For any fixed $p \in M$, $v \in T_pM$,

$\exists$ "!" geodesic $c_{p,v} : (-\varepsilon, \varepsilon) \to M$ s.t. $c(0) = p$, $c'(0) = v$



(M,g)

Exponential Map



$$\exp_p : \underset{0 \in}{\mathcal{U}} \subseteq T_pM \longrightarrow M$$

$$\exp_p(v) := C_{p,v}(1)$$

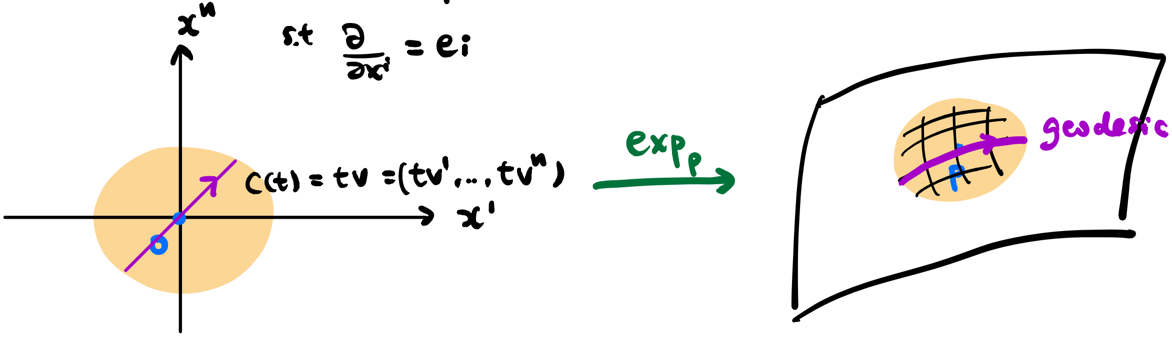
- $\exp_p(0) = p$
- $d(\exp_p)_0 = id_{T_pM}$

$\overset{\text{I.F.T.}}{\Longrightarrow}$ $\exp_p$ is a local diffeomorphism near 0 to a nbd. of p.

$\rightsquigarrow$ local coordinate system near p called "Geodesic Normal Coordinates"

$\langle , \rangle_p$ $\boxed{T_pM}$ Fix O.N.B $e_1, \dots, e_n$ and choose coord $(x^1, \dots, x^n)$ on T_pM s.t $\frac{\partial}{\partial x^i} = e_i$

$\boxed{M, g}$



Prop: In geodesic normal coord. at $p \in M$,

$$g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \Gamma_{ij}^k(0) = 0$$

(i.e. (M^n, g) "=" $(\mathbb{R}^n, g_{Eucl.})$ at any pt. $\hookrightarrow$ 1^{st} order information at a pt is NOT "Geometric" (indep. of choice of coord.))

Proof: $g_{ij}(0) = \langle \partial_i, \partial_j \rangle_0 = \delta_{ij}$ by construction.

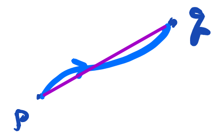
radial lines from 0 $c(t) = tv$ corr. to geodesics on (M, g)

$$\Rightarrow \underbrace{\frac{d^2c^k}{dt^2}}_{=0} + \Gamma_{ij}^k \underbrace{\frac{dc^i}{dt}}_{v^i} \underbrace{\frac{dc^j}{dt}}_{v^j} = 0 \Rightarrow \Gamma_{ij}^k(0) v^i v^j = 0 \;\; \forall v \Rightarrow \Gamma_{ij}^k(0) = 0$$

$\blacksquare$

<u>Recall:</u> "geodesics" ≈ "straight lines"

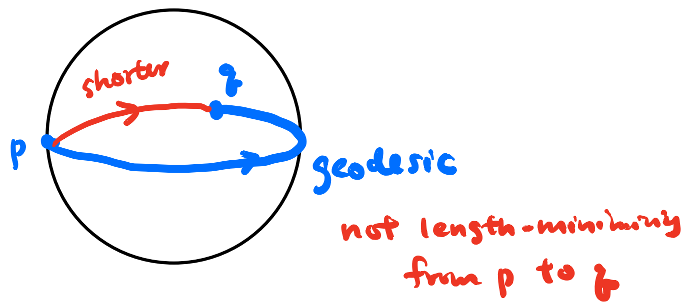
acceleration $= 0$ — length-minimizing curves



We will see that geodesics are "locally" length-minimizing.

<u>E.g.)</u> (S^2, g_{round})

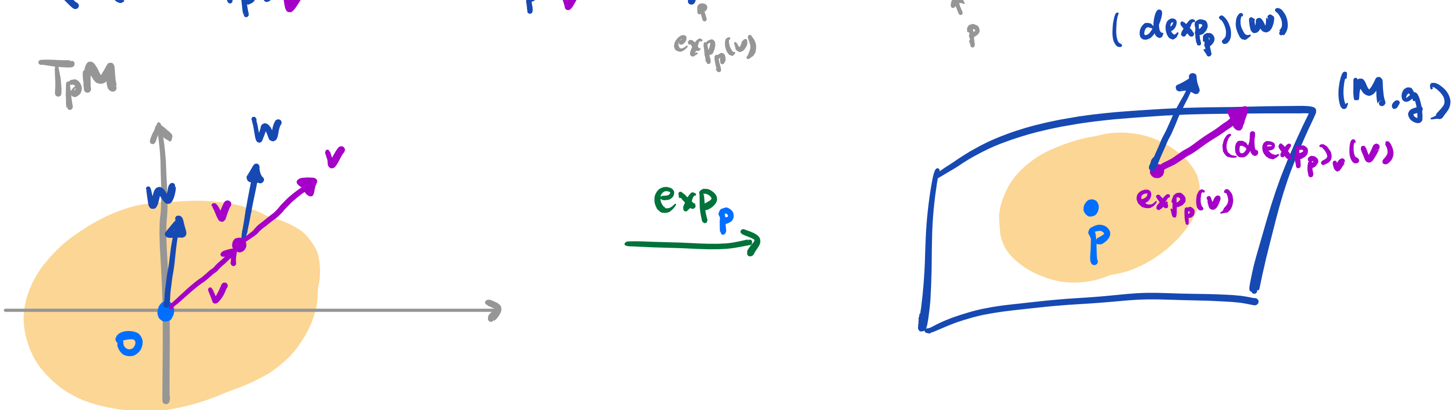
"Long" geodesics are not necessarily minimizing.



<u>Gauss lemma:</u> Let $p \in M$, $v \in T_pM$ s.t. $\exp_p(v)$ is defined.

Then, $\forall\, w \in T_pM \;(\cong T_v(T_pM))$,

$$\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle_{\exp_p(v)} = \langle v, w \rangle_p$$



<u>Proof:</u> <u>Case 1</u>: $w = v$

$\because\; t \longmapsto \exp_p(tv)$ geodesic on M

$\Rightarrow$ constant speed $= \|v\|$ at $t=0$.

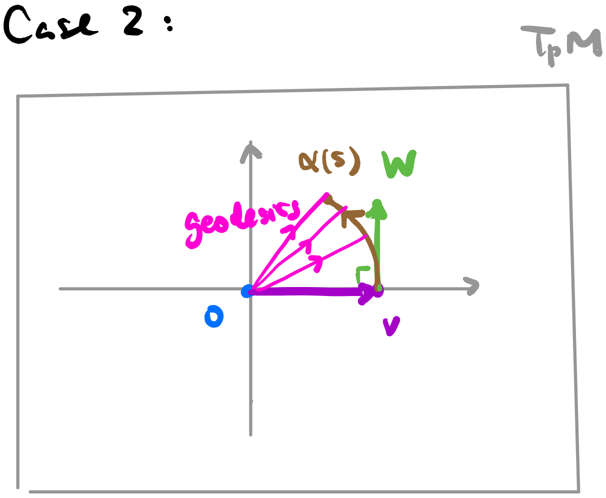
<u>Case 2</u>: $w \perp v$ w.r.t. $\langle , \rangle_p$.

It suffices to show

$$(d\exp_p)_v(w) \perp (d\exp_p)_v(v)$$

w.r.t. $\langle , \rangle_{\exp_p(v)}$

Case 2:



$\alpha(s)$: a curve on T_pM

s.t. $\alpha(0) = v$; $\alpha'(0) = w$

and $\|\alpha(s)\| \equiv \|v\|$

$\rightsquigarrow$ get a 1-parameter family of geodesics by

$$f_s(t) := \exp_p(t\alpha(s)).$$

<u>Note</u>: $f_s(\cdot)$ is a geodesic for each s.

<u>Observe</u>:

$$(d\exp_p)_v(v) = \left.\frac{\partial f}{\partial t}\right|_{t=1,\, s=0}$$

$$(d\exp_p)_v(w) = \left.\frac{\partial f}{\partial s}\right|_{t=1,\, s=0}$$

Consider

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle \overset{\text{metric compatible}}{=} \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \cancelto{0}{\nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial t}}\right\rangle$$

$$\overset{\text{torsion free}}{=} \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle \overset{\text{metric compatible}}{=} \frac{1}{2}\frac{\partial}{\partial s}\underbrace{\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle}_{\substack{\equiv \text{ const. in } s \\ \because \|\alpha(s)\| \equiv \|v\|.}} = 0$$

So, $\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle$ is indep. of t.

$$\left.\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle\right|_{t=1,\, s=0} = \left.\left\langle \underbrace{\frac{\partial f}{\partial s}}_{\substack{\| \\ 0}\ \text{at } t=0}, \frac{\partial f}{\partial t}\right\rangle\right|_{t=0,\, s=0} = 0$$

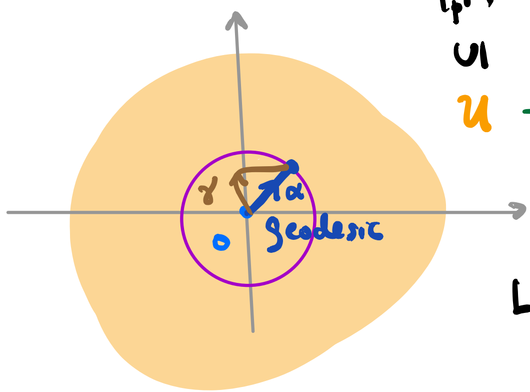
$\square$

Gauss Lemma, in geodesic normal coord., using polar coord (r, θ)

$$\Rightarrow \quad g_{rr} \equiv 1 \quad \text{and} \quad g_{r\theta} \equiv 0$$

$\Longrightarrow$ geodesics are locally length-minimizing.

Why?



$$\begin{array}{c} T_pM \\ \cup\!| \\ \mathcal{U} \end{array} \xrightarrow[\text{diffeo.}]{\exp_p} \exp_p \mathcal{U} \subseteq M$$

$$\text{Length}(\gamma) := \int_0^1 \sqrt{\langle \gamma', \gamma' \rangle}\, dt$$

$$= \int_0^1 \sqrt{\underbrace{g_{rr}}_{\substack{= \\ 1}\ \text{Gauss lemma}}(r')^2 + \underbrace{g_{\theta\theta}(\theta')^2}_{\geq 0} + 2\underbrace{g_{r\theta}}_{\substack{= \\ 0}\ \text{Gauss lemma}} r'\theta'}\, dt$$

$$\geq \int_0^1 |r'|\, dt \geq \int_0^1 r'\, dt$$

$$= r(1) - r(0) = \text{Length}(\alpha)$$